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Darboux transformations and supersymmetry for the generalized Schrödinger equations in (1+1) dimensions

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Abstract

We study intertwining relations, supersymmetry and Darboux transformations for generalized Schrödinger equations in (1+1) dimensions. We obtain intertwiners in an explicit form by means of which we construct arbitrary-order Darboux transformations for our class of equations. We develop a corresponding supersymmetric formulation and prove equivalence of the Darboux transformations with the supersymmetry formalism. Finally, we show that our Darboux transformations can also be constructed by means of point transformations, avoiding the use of intertwiners.

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1. Introduction

The Darboux transformation is a method for solving differential equations which has become popular especially in quantum mechanics, where it is also known as a supersymmetric factorization method. Mathematically, the Darboux transformation is a linear differential operator that maps solutions of a differential equation onto solutions of another differential equation. Typically, these equations are of a similar form, but differ in a nonconstant parameter only, such as in the case of two Schrödinger equations that are the same up to their potential. While in principle the Darboux transformation works similarly to Lie symmetry methods, its structure is essentially different, as it does not involve any change of coordinates. Since originally [1] the Darboux transformation was constructed for ordinary differential equations only, in the context of quantum mechanics it was found to be applicable to the stationary Schrödinger equation [2]. Much later it was found that the Darboux transformation is equivalent to the supersymmetry formalism, see the reviews in [3, 4]. Darboux transformations have been constructed for a variety of linear and nonlinear equations [5, 6]. In fact, here all classical nonlinear equations are considered, such as the nonlinear Schrödinger equation, the Korteweg–de Vries equation, the sine-Gordon equation and many more. Among the linear equations to which the Darboux transformations are applicable, there is a class of linear generalizations of the Schrödinger equation such as the Schrödinger equations with position-dependent mass [7–10], Schrödinger equations with weighted energy [11] and generalized Schrödinger equations with position-dependent mass and weighted energy [12]. All the above-mentioned cases are related to time-independent Hamiltonians. As is well known, everything in nature changes in time and scientists need to know how physical processes evolve with time.

In recent years, great interest has been arisen in developing Darboux transformations for time-dependent Schrödinger equations [5, 13–16]. As far as we know the first attempts to construct the Darboux transformations were made in [5]. These considerations can be seen as special cases of a certain generalized, nonstationary equation of Schrödinger-type, which admits a generalized Darboux transformation that we constructed explicitly in a recent paper [17]. This construction of ours left two important questions open, the answers to which will be our focus in the present paper. The first open question concerns the relation between our Darboux transformation and a possible supersymmetry formalism. It is well known that the conventional Schrödinger equation and its self-adjoint generalizations (e.g. the position-dependent mass equation) allow a supersymmetric formulation, which is equivalent to their respective Darboux transformation. However, in the case of our generalized, time-dependent equation this is not clear at all. The second question concerns the method of construction that we applied in [17]: in contrast to the common approach for constructing the Darboux transformations, we did not use intertwining relations, but instead made use of point-canonical transformations. Thus, the question remains whether our generalized Darboux transformations could also be obtained by evaluating intertwining relations and in what way the two construction methods—with and without intertwining relations—are related. Both of these questions will be answered in the present work.

The paper is organized as follows. In the following section, we evaluate the intertwining relation for the first-order Darboux transformation and derive the intertwiner, as well as the transformed potential. Section 3 is devoted to the reality condition which guarantees that the transformed potential is a real-valued function. In section 4, we develop the supersymmetry formalism for our generalized Schrödinger equation and show its equivalence with the generalized Darboux transformation. Section 5 is devoted to the higher order Darboux transformations which we construct by evaluating the corresponding intertwining relation. Furthermore, we will show that the two construction methods—using intertwining relations and point-canonical transformations—are equivalent and therefore lead to the same results. Finally, in section 6, we give an explicit example in order to illustrate our method.

2. First-order Darboux transformation

Consider the following generalized, time-dependent Schrödinger equation in (1+1) dimensions and units $\hbar^2/2 = 1$:

$$\mathrm{i}h\psi_t = -\left[\partial_x\left(\frac{1}{m}\right)\partial_x\right]\psi + v\psi, \tag{1}$$

where the index and the symbol ∂ denote the partial differentiation, $m = m(x, t)$ stands for the particle's effective mass, $h = h(x, t)$ and $v = v(x, t)$ denote the potentials, and $\psi = \psi(x, t)$ is the solution. This equation can be rewritten as

$$\mathrm{i}\psi_t = \mathcal{H}\psi, \qquad \mathcal{H} = -\frac{1}{h}\left[\partial_x\left(\frac{1}{m}\right)\partial_x\right] + \frac{v}{h}. \tag{2}$$

Introduce the transformation operator $\mathcal{L}$ that acts on the solutions of (1) as follows:

$$\mathcal{L}(\mathrm{i}\partial_t - \mathcal{H}) = (\mathrm{i}\partial_t - \widetilde{\mathcal{H}})\mathcal{L}, \tag{3}$$

where

$$\widetilde{\mathcal{H}} = -\frac{1}{h}\left[\partial_x\left(\frac{1}{m}\right)\partial_x\right] + \frac{\widetilde{v}}{h}. \tag{4}$$

Equation (3) and the operator $\mathcal{L}$ are called the intertwining relation and the intertwiner, respectively. The operator $\mathcal{L}$ transforms any solution ψ of (1) into a solution

$$\widetilde{\psi} = \mathcal{L}\psi \tag{5}$$

of the transformed Schrödinger equation

$$(\mathrm{i}\partial_t - \widetilde{\mathcal{H}})\widetilde{\psi} = 0. \tag{6}$$

We search for the intertwiner in the form of a linear, first-order differential operator

$$\mathcal{L} = A + B\partial_x, \tag{7}$$

where $A = A(x,t)$ and $B = B(x,t)$ are to be determined, such that $\mathcal{L}$ satisfies (3). By substituting (7) and the explicit form of the Hamiltonians $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ into the intertwining relation (3) and applying it to the solution ψ of (1), we get

$$\mathcal{L}\left[\mathrm{i}\partial_t + \frac{1}{hm}\partial_{xx} + \frac{1}{h}\left(\frac{1}{m}\right)_x\partial_x - \frac{v}{h}\right]\psi = \left[\mathrm{i}\partial_t + \frac{1}{hm}\partial_{xx} + \frac{1}{h}\left(\frac{1}{m}\right)_x\partial_x - \frac{\widetilde{v}}{h}\right]\mathcal{L}\psi.$$

Assuming linear independence of ψ and its partial derivatives, we collect their respective coefficients and equal them to zero. This leads to the following system of equations for the functions A, B and $\widetilde{v}$:

$$\frac{2}{hm}B_x = \left(\frac{1}{hm}\right)_x B, \tag{8}$$

$$\mathrm{i}B_t + \frac{2}{hm}A_x + \frac{1}{hm}B_{xx} + \frac{1}{h}\left(\frac{1}{m}\right)_x B_x - B\left[\frac{1}{h}\left(\frac{1}{m}\right)_x\right]_x = \frac{1}{h}(\widetilde{v} - v)B, \tag{9}$$

$$\mathrm{i}A_t + \frac{1}{hm}A_{xx} + \left(\frac{v}{h}\right)_x B + \frac{1}{h}\left(\frac{1}{m}\right)_x A_x = \frac{1}{h}(\widetilde{v} - v)A. \tag{10}$$

Condition (8) determines B:

$$\frac{2B_x}{B} = -\frac{(hm)_x}{hm} \rightsquigarrow B = \frac{\beta}{\sqrt{hm}}, \tag{11}$$

where $\beta = \beta(t)$ is an arbitrary, purely time-dependent constant of integration. Equations (9) and (10) enable us to determine the potential $\widetilde{v}$ and the function A. For this, let us multiply equation (9) with A and equation (10) with B. Then, the left-hand sides of (9) and (10) become the same and we can set them equal to each other:

$$\begin{aligned}\mathrm{i}AB_t + \frac{2}{hm}AA_x + \frac{1}{hm}AB_{xx} + \frac{1}{h}\left(\frac{1}{m}\right)_x AB_x - \left[\frac{1}{h}\left(\frac{1}{m}\right)_x\right]_x AB \\ -\mathrm{i}A_tB - \frac{1}{hm}A_{xx}B - \left(\frac{v}{h}\right)_x B^2 - \frac{1}{h}\left(\frac{1}{m}\right)_x A_xB = 0.\end{aligned} \tag{12}$$

We will now solve this equation with respect to A. To this end, we introduce a new auxiliary function K defined by $A = BK$. Taking into account (11) and

$$\frac{2B_x}{B} = hm\left(\frac{1}{hm}\right)_x, \qquad \frac{B_{xx}}{B} = -\frac{(hm)_{xx}}{2hm} + \frac{3}{4}\left[\frac{(hm)_x}{hm}\right]^2,$$

equation (12) is transformed into an equation for K only:

$$\mathrm{i}K_t = \left[-\frac{K_x}{hm} + \frac{|K|^2}{hm} - \frac{1}{h}\left(\frac{1}{m}\right)_x K - \frac{v}{h}\right]_x. \tag{13}$$

This is a time-dependent Riccati equation that can be linearized and integrated by introducing a new function $\mathcal{U} = \mathcal{U}(x,t)$:

$$K = -\frac{\mathcal{U}_x}{\mathcal{U}}. \tag{14}$$

Assuming that $\mathcal{U}$ is twice continuously differentiable, implying $\mathcal{U}_{xt} = \mathcal{U}_{tx}$, we substitute (14) in (13) and get

$$\left[\mathrm{i}\frac{\mathcal{U}_t}{\mathcal{U}} + \frac{1}{hm}\frac{\mathcal{U}_{xx}}{\mathcal{U}} + \frac{1}{h}\left(\frac{1}{m}\right)_x\frac{\mathcal{U}_x}{\mathcal{U}} - \frac{v}{h}\right]_x = 0. \tag{15}$$

Clearly, this equation holds if the expression in brackets does not depend on x. We integrate on both sides and multiply with $\mathcal{U}$:

$$\mathrm{i}\mathcal{U}_t + \frac{1}{hm}\mathcal{U}_{xx} + \frac{1}{h}\left(\frac{1}{m}\right)_x\mathcal{U}_x - \frac{v}{h}\mathcal{U} = C\mathcal{U}, \tag{16}$$

where $C = C(t)$ is a purely time-dependent constant of integration. Equation (16) is identical to the initial equation (1) for $C = 0$. However, setting C to zero is not a restriction, since solutions to (16) with $C \neq 0$ and $C = 0$ differ from each other only by a purely time-dependent factor, which cancels out in (14). Once $\mathcal{U}$ is given, one can then evaluate the function K via (14), which in turn determines A by means of $A = BK$ and (11), that is,

$$A = -\frac{\beta}{\sqrt{hm}}\log(\mathcal{U})_x. \tag{17}$$

Let us point out here that the notation $\log(\mathcal{U})_x$ refers to a derivative of the logarithm and not only to a derivative of its argument. Having found A from (12), one can get the equation for the new potential $\widetilde{v}$ by solving (9) for $\widetilde{v}$:

$$\widetilde{v} = v + \frac{B_{xx}}{mB} + \frac{2A_x}{mB} + \left(\frac{1}{m}\right)_x\frac{B_x}{B} - h\left[\frac{1}{h}\left(\frac{1}{m}\right)_x\right]_x + \mathrm{i}h\frac{B_t}{B}. \tag{18}$$

By insertion of expressions (11) and (17) for B and A, respectively, we obtain the explicit form of the transformed potential (18). Furthermore, we can construct the intertwiner $\mathcal{L}$ and the transformed solution $\widetilde{\psi}$ from (7) and (5), respectively:

$$\widetilde{v} = v + \mathrm{i}h\left[\frac{\beta_t}{\beta} - \frac{1}{2}\log(hm)_t\right] - 2\sqrt{\frac{h}{m}}\left[\frac{(u_1)_x}{u_1\sqrt{hm}}\right]_x - \sqrt{\frac{h}{m}}\left[\frac{1}{h}\left(\sqrt{\frac{h}{m}}\right)_x\right]_x, \tag{19}$$

$$\mathcal{L} = \frac{\beta}{\sqrt{hm}}(\partial_x + K) = \frac{\beta}{\sqrt{hm}}[\partial_x - \log(\mathcal{U})_x], \tag{20}$$

$$\tilde{\psi} = \mathcal{L}\psi = \frac{\beta}{\sqrt{hm}}[\partial_x - \log(\mathcal{U})_x]\psi. \tag{21}$$

The transformation function $\mathcal{U}$ defines the transformation operator $\mathcal{L}$, the new potential $\tilde{v}$ and the corresponding solutions $\tilde{\psi}$. The new potential depends not only on the potential v, but also on the additional potentials m and h. The form of the Darboux operator and of the transformed potential as given above coincides with our previous results [17], which were obtained without making use of the intertwining relations, but by means of point canonical transformations. In other words, the results of this section show that both construction methods are equivalent. We will prove in section 4 that this also holds for the higher order Darboux transformations.

3. Reality condition

Let us find out conditions under which the constructed potential $\tilde{v}$ will be real if the initial potentials v, h and m are real. To this end, we take the transformed potential $\widetilde{v}$ in its form (19) and extract its imaginary part. Since the function β can be complex and appears within a logarithm in (19), let us first determine its real and imaginary parts. Write β in polar coordinates as

$$\beta = \beta_1 \exp(\mathrm{i}\beta_2),$$

where $\beta_1 = \beta_1(t)$ and $\beta_2 = \beta_2(t)$ denote the absolute value and the argument of β, respectively. We obtain

$$\begin{aligned} \log(\beta)_t &= \frac{\beta_t}{\beta} \\ &= \frac{(\beta_1 \exp(\mathrm{i}\beta_2))_t}{\beta_1 \exp(\mathrm{i}\beta_2)} \\ &= \log(\beta_1)_t + \mathrm{i}(\beta_2)_t. \end{aligned} \tag{22}$$

Thus, extracting the imaginary part of the transformed potential (19) gives

$$\mathrm{Im}(\widetilde{v}) = h\left[\log(\beta_1)_t - \frac{1}{2}\log(hm)_t\right] + 2\sqrt{\frac{h}{m}}\left[\frac{\mathrm{Im}(K)}{\sqrt{hm}}\right]_x.$$

If this expression vanishes, then the transformed potential $\widetilde{v}$ is real valued, that is

$$\log(\beta_1)_t = \frac{1}{2}\log(hm)_t - 2\sqrt{\frac{1}{hm}}\left[\frac{\mathrm{Im}(K)}{\sqrt{hm}}\right]_x.$$

On employing definition (14) of the function K, we obtain

$$\begin{aligned} \log(\beta_1)_t &= \frac{1}{2}\log(hm)_t + 2\sqrt{\frac{1}{hm}}\left[\frac{\mathrm{Im}\left(\frac{\mathcal{U}_x}{\mathcal{U}}\right)}{\sqrt{hm}}\right]_x \\ &= \frac{1}{2}\log(hm)_t + 2\sqrt{\frac{1}{hm}}\left[\frac{\mathrm{Im}\left(\log(\mathcal{U})_x\right)}{\sqrt{hm}}\right]_x \\ &= \frac{1}{2}\log(hm)_t - \mathrm{i}\sqrt{\frac{1}{hm}}\left[\frac{\log(\mathcal{U})_x - \log(\mathcal{U}^*)_x}{\sqrt{hm}}\right]_x \\ &= \frac{1}{2}\log(hm)_t - \mathrm{i}\sqrt{\frac{1}{hm}}\left[\sqrt{\frac{1}{hm}}\log\left(\frac{\mathcal{U}}{\mathcal{U}^*}\right)_x\right]_x. \end{aligned} \tag{23}$$

This is the reality condition for our transformed potential (19). In the conventional case, where h and m are constants, condition (23) becomes

$$\log(\beta_1)_t = -\frac{\mathrm{i}}{hm}\log\left(\frac{\mathcal{U}}{\mathcal{U}^*}\right)_{xx}.$$

This is fulfilled if the right-hand side does not depend on x, that is if

$$\log\left(\frac{\mathcal{U}}{\mathcal{U}^*}\right)_{xxx} = 0,$$

which coincides with the findings in [13]. We can solve (23) for the function β_1, giving

$$\beta_1 = \sqrt{hm}\exp\left[-\mathrm{i}\int\sqrt{\frac{1}{hm}}\left[\sqrt{\frac{1}{hm}}\log\left(\frac{\mathcal{U}}{\mathcal{U}^*}\right)_x\right]_x \mathrm{d}t\right]. \tag{24}$$

Note that the left-hand side of this equality does not depend on x, while the right-hand side does. This means that condition (24) cannot always be fulfilled, except if its right-hand side is purely time dependent. Now suppose that the reality condition (23) or, equivalently, (24) is fulfilled. We substitute into the transformed potential (19), which then takes the following form:

$$\begin{aligned}
\widetilde{v} &= v - h(\beta_2)_t + 2\sqrt{\frac{h}{m}}\left[\frac{\mathrm{Re}(K)}{\sqrt{hm}}\right]_x - \sqrt{\frac{h}{m}}\left[\frac{1}{h}\left(\sqrt{\frac{h}{m}}\right)_x\right]_x \\
&= v - h(\beta_2)_t - 2\sqrt{\frac{h}{m}}\left[\frac{\mathrm{Re}\left(\frac{\mathcal{U}_x}{\mathcal{U}}\right)}{\sqrt{hm}}\right]_x - \sqrt{\frac{h}{m}}\left[\frac{1}{h}\left(\sqrt{\frac{h}{m}}\right)_x\right]_x \\
&= v - h(\beta_2)_t - 2\sqrt{\frac{h}{m}}\left[\frac{\mathrm{Re}\,(\log(\mathcal{U})_x)}{\sqrt{hm}}\right]_x - \sqrt{\frac{h}{m}}\left[\frac{1}{h}\left(\sqrt{\frac{h}{m}}\right)_x\right]_x \\
&= v - h(\beta_2)_t - \sqrt{\frac{h}{m}}\left[\frac{\log(\mathcal{U})_x + \log(\mathcal{U}^*)_x}{\sqrt{hm}}\right]_x - \sqrt{\frac{h}{m}}\left[\frac{1}{h}\left(\sqrt{\frac{h}{m}}\right)_x\right]_x \\
&= v - h(\beta_2)_t - \sqrt{\frac{h}{m}}\left[\frac{1}{\sqrt{hm}}\log(|\mathcal{U}|^2)_x\right]_x - \sqrt{\frac{h}{m}}\left[\frac{1}{h}\left(\sqrt{\frac{h}{m}}\right)_x\right]_x.
\end{aligned} \tag{25}$$

This expression is clearly real valued and compatible with the conventional case, where h and m are constants: we get for the transformed potential (25)

$$\widetilde{v} = v - h(\beta_2)_t - \frac{1}{m}\log(|\mathcal{U}|^2)_{xx},$$

which coincides with the result obtained in [13], if we set the arbitrary phase β_2 to zero.

4. Supersymmetry

We will now develop the formalism of supersymmetry for our generalized Schrödinger equation (1) and show that this formalism is equivalent to the Darboux transformation. To this end, define the operation of conjugation as follows: $(AB)^+ = B^+A^+$ with respect to which the Schrödinger operator $\mathrm{i}\partial_t - \mathcal{H}$ is self-adjoint

$$(\mathrm{i}\partial_t - \mathcal{H})^+ = \mathrm{i}\partial_t - \mathcal{H}.$$

On taking the adjoint on both sides of the intertwining relation (3), we obtain

$$(\mathrm{i}\partial_t - \mathcal{H})\mathcal{L}^+ = \mathcal{L}^+(\mathrm{i}\partial_t - \widetilde{\mathcal{H}}). \tag{26}$$

The operator $\mathcal{L}^+$ adjoint to $\mathcal{L}$, as given in (20), is determined as

$$\mathcal{L}^+ = \left[\frac{1}{\sqrt{hm}}\left(-\partial_x + K^+\right) - \frac{1}{h}\partial_x\sqrt{\frac{h}{m}}\right]\beta^+. \tag{27}$$

Now, let ψ and $\tilde{\psi}$ be the solutions to the Schrödinger equations associated with the operators $\mathrm{i}\partial_t - \mathcal{H}$ and $\mathrm{i}\partial_t - \widetilde{\mathcal{H}}$, respectively (recall definition (4) of $\widetilde{\mathcal{H}}$). These Schrödinger equations can then be written as one single matrix equation in the form

$$\left[\begin{pmatrix} \mathrm{i}\partial_t & 0 \\ 0 & \mathrm{i}\partial_t \end{pmatrix} - \begin{pmatrix} \mathcal{H} & 0 \\ 0 & \widetilde{\mathcal{H}} \end{pmatrix}\right]\begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix} = 0. \tag{28}$$

On defining $H_m = \mathrm{diag}(\mathcal{H}, \widetilde{\mathcal{H}})$ and $\Psi = (\psi, \tilde{\psi})^T$, the above matrix time-dependent Schrödinger equation (28) can be written as

$$[\mathrm{i}\partial_t - H_m]\Psi = 0. \tag{29}$$

Similar to the case of the constant mass [14], we now define two supercharge operators Q, Q^+ as follows:

$$Q = \begin{pmatrix} 0 & 0 \\ \mathcal{L} & 0 \end{pmatrix}, \qquad Q^+ = \begin{pmatrix} 0 & \mathcal{L}^+ \\ 0 & 0 \end{pmatrix}, \tag{30}$$

where $\mathcal{L}$ and $\mathcal{L}^+$ are the operators given by (20) and (27), respectively. One can show that the matrix Hamiltonian H_m, as given in (29), satisfies the following conditions:

$$\left\{\begin{matrix} \{Q, Q\} = \{Q^+, Q^+\} = 0 \\ [Q, \mathrm{i}\partial_t - H_m] = [\mathrm{i}\partial_t - H_m, Q] = 0 \\ [\mathrm{i}\partial_t - H_m, Q^+] = [Q^+, \mathrm{i}\partial_t - H_m] \end{matrix}\right\}, \tag{31}$$

where $\{\cdot, \cdot\}$ and $[\cdot, \cdot]$ are the anticommutator and commutator, respectively. The first line of the latter set is trivially fulfilled, because the matrices in (30) are nilpotent. The conditions in the second and third lines of (31) are precisely our intertwining relations (3) and (26). Now, let us consider the complementing relations of the supersymmetric algebra, that is the anticommutators $\{Q, Q^+\}$ and $\{Q^+, Q\}$. For this, we calculate the operators $G = \mathcal{L}^+\mathcal{L}$ and $\tilde{G} = \mathcal{L}\mathcal{L}^+$, and consider the connections of them with our Hamiltonians $\mathcal{H}$ and $\widetilde{\mathcal{H}}$. By using (20) and (27), we arrive after some algebraic transformations at

$$G = \mathcal{L}^+\mathcal{L} = |\beta|^2\left[-\frac{1}{hm}\partial_{xx} - \frac{1}{h}\left(\frac{1}{m}\right)_x\partial_x + \frac{1}{hm}(|K|^2 - K_x) - \frac{1}{h}\left(\frac{1}{m}\right)_x K\right], \tag{32}$$

$$\tilde{G} = \mathcal{L}\mathcal{L}^+ = |\beta|^2\left\{-\frac{1}{hm}\partial_{xx} - \frac{1}{h}\left(\frac{1}{m}\right)_x\partial_x + \frac{1}{hm}(|K|^2 + K_x) + \frac{1}{m}\left(\frac{1}{h}\right)_x K \right.$$
$$\left. - \frac{1}{\sqrt{hm}}\left[\frac{1}{h}\left(\sqrt{\frac{h}{m}}\right)_x\right]_x\right\}. \tag{33}$$

The difference of the last two expressions yields

$$\widetilde{G} - G = 2|\beta|^2\sqrt{\frac{1}{hm}}\left(\frac{K}{\sqrt{mh}}\right)_x - |\beta|^2\sqrt{\frac{1}{hm}}\left[\frac{1}{h}\left(\sqrt{\frac{h}{m}}\right)_x\right]_x. \tag{34}$$

Next, we consider the diagonal matrix $\mathcal{G} = \mathrm{diag}(G, \tilde{G})$, which is a symmetry operator of our matrix equation (29). Note that in the stationary case the diagonal components of $\mathcal{G}$ can be the Hamiltonians $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, for details see [13]. The supercharges Q, Q^+ and the symmetry operator $\mathcal{G}$ generate the simplest superalgebra that can be written in a standard form

$$Q^2 = (Q^+)^2 = 0, \qquad [Q, \mathcal{G}] = [Q^+, \mathcal{G}] = 0, \qquad \{Q, Q^+\} = \{Q^+, Q\} = \mathcal{G}. \tag{35}$$

It is useful to compare relations (31) and (35). The first equations in (31) and (35) coincide. The intertwining relations are different. They are standard for the operators G and $\tilde{G}$, that is

$$\widetilde{G}\mathcal{L} - \mathcal{L}G = 0, \qquad G\mathcal{L}^+ - \mathcal{L}^+\widetilde{G} = 0.$$

But the intertwining relations for the operators $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are nonstandard. From (3) and (26) it follows

$$\widetilde{\mathcal{H}}\mathcal{L} - \mathcal{L}\mathcal{H} = \mathrm{i}\mathcal{L}_t - \mathrm{i}B\partial_{xt}, \qquad \mathcal{L}^+\widetilde{\mathcal{H}} - \mathcal{H}\mathcal{L}^+ = -\mathrm{i}\mathcal{L}_t^+ + \mathrm{i}B^*\partial_{xt}. \tag{36}$$

By comparing the intertwining relations for the elements of H_m and $\mathcal{G}$ and by taking into account the third equation of (35), one can deduce the connection between operators $\mathcal{G}$ and H_m as follows. Let $\lambda_1 = \lambda_1(x,t)$ and $\lambda_2 = \lambda_2(x,t)$ be the functions that are to be determined and define $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$. Then

$$\mathcal{G} = \begin{pmatrix} G & 0 \\ 0 & \widetilde{G} \end{pmatrix} = \begin{pmatrix} \mathcal{H} - \lambda_1 & 0 \\ 0 & \widetilde{\mathcal{H}} - \lambda_2 \end{pmatrix} = H_m - \Lambda, \tag{37}$$

where we take $|\beta|^2 = 1$. In components, the latter equality reads

$$\mathcal{H} = G + \lambda_1 = \mathcal{L}^+\mathcal{L} + \lambda_1, \tag{38}$$

$$\widetilde{\mathcal{H}} = \widetilde{G} + \lambda_2 = \mathcal{L}\mathcal{L}^+ + \lambda_2, \tag{39}$$

and in an equivalent form

$$H_m = \{Q^+, Q\} + \Lambda. \tag{40}$$

We now determine the function λ_1. To this end, note that using (13), (14) and (16) we can express v/h in the form

$$\frac{v}{h} = \frac{1}{hm}(|K|^2 - K_x) - \frac{1}{h}\left(\frac{1}{m}\right)_x K + \mathrm{i}\frac{\mathcal{U}_t}{\mathcal{U}} - C. \tag{41}$$

Recall that $C = C(t)$ is an arbitrary function that appeared as a constant of integration in (16). We now find λ_1 from (38) by substitution of (32) and comparison with (41):

$$\lambda_1 = \mathrm{i}\frac{\mathcal{U}_t}{\mathcal{U}} - C.$$

It remains to determine the function λ_2 that appeared in (37). We find this function from the potential difference (19), which after the substitution of (41) takes the following form:

$$\frac{\tilde{v}}{h} = \frac{1}{hm}(|K|^2 + K_x) + \frac{1}{m}\left(\frac{1}{h}\right)_x K - \frac{1}{\sqrt{hm}}\left[\frac{1}{h}\left(\sqrt{\frac{h}{m}}\right)_x\right]_x + \lambda_1 + \mathrm{i}\frac{B_t}{B}. \tag{42}$$

Similar to the case of λ_1, we substitute (33) into (39) and obtain

$$\lambda_2 = \lambda_1 + \mathrm{i}\frac{B_t}{B}.$$

This means that two Hamiltonians $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ can be factorized as is given in (38) and (39) or in the form (40), supplementing the supersymmetric relations (31) for the generalized time-dependent Hamiltonian. The fact that the factorization matrix Λ depends on spatial and time variables through the functions $\mathcal{U} = \mathcal{U}(x,t)$ and $B = B(x,t)$ is nonstandard. In a particular case, when the potentials m and h do not depend on time, Λ can be written as $\Lambda = \lambda_1 I$, where I is the identity matrix. Moreover, if the transformation function $\mathcal{U}$ can be presented in a factorized form like $\mathcal{U}(x,t) = F(x)S(t)$, then λ_1 becomes independent of the spatial variable.

Note that as soon as the initial Hamiltonian $\mathcal{H}$ is presented in the factorized form (38), one can obtain its supersymmetric partner in a factorized form too. Indeed, multiplying equation (38) from the left by $\mathcal{L}$ and taking into account the first intertwining relation from (36), we get

$$\mathcal{L}\mathcal{H}\psi = \mathcal{L}(\mathcal{L}^+\mathcal{L} + \lambda_1)\psi = \widetilde{\mathcal{H}}\mathcal{L}\psi + \mathrm{i}B\psi_{tx} - \mathrm{i}\mathcal{L}_t\psi. \tag{43}$$

This enables us to express $\widetilde{\mathcal{H}}$ in terms of $\mathcal{L}$ and $\mathcal{L}^+$. Let us calculate

$$\mathrm{i}\mathcal{L}_t\psi = \mathrm{i}B_t(\partial_x + K)\psi + \mathrm{i}B(\partial_x + K)_t\psi = \mathrm{i}\frac{B_t}{B}\mathcal{L}\psi + B\left(\mathrm{i}\partial_t - \mathrm{i}\frac{\mathcal{U}_t}{\mathcal{U}}\right)_x\psi = \mathrm{i}\frac{B_t}{B}\mathcal{L}\psi.$$

In the last step, we took into account that $\mathrm{i}\psi_t = \mathcal{H}\psi$ and $\mathrm{i}\mathcal{U}_t/\mathcal{U} = \mathcal{H}$. Now, we compute

$$\mathcal{L}\lambda_1\psi = B(\partial_x + K)\lambda_1\psi = B(\lambda_1)_x\psi + \lambda_1\mathcal{L}\psi.$$

Using the last equations in (43) and taking into account that

$$\mathrm{i}B(\lambda_1)_x\psi - \mathrm{i}B\psi_{tx} = B\left(\mathrm{i}\frac{\mathcal{U}_t}{\mathcal{U}}\psi - \mathrm{i}\psi_t\right)_x = 0,$$

we arrive at

$$\widetilde{\mathcal{H}} = \mathcal{L}\mathcal{L}^+ + \mathrm{i}\frac{B_t}{B} + \lambda_1 \tag{44}$$

that coincides with (39).

In summary, we obtained the explicit forms of the supersymmetric partner Hamiltonians $\mathcal{H}$ and $\widetilde{\mathcal{H}}$. Hamiltonians (38) and (39) are compatible with their definitions (2) and (4), respectively, if $|\beta|^2 = 1$ and if the transformed potentials v and $\tilde{v}$ are given by (41) and (42). Finally, taking the difference of the factorized Hamiltonians (38) and (39) gives the potential difference (19) that we obtained for our Darboux transformation. Hence, the Darboux transformation is equivalent to the supersymmetry formalism.

5. Higher order Darboux transformation

In this section, we will derive nth-order Darboux transformations for the generalized Schrödinger equation (1). In contrast to our former work [17], here we will use intertwiners for the construction of the Darboux transformation. It will then turn out that our former method—construction via point canonical transformations—is equivalent to solving the intertwining relation. In principle, one would have to go back to an intertwining relation (3) with generalized Hamiltonians (2), (4), and solve it for an nth-order differential operator L, which would imply large and involved calculations. Here, we will pursue a different approach for solving the intertwining relation. We start from the conventional Schrödinger Darboux transformation and its well-known intertwining relation, which—by means of a point transformation—we take into an intertwining relation for the generalized case. The construction of our nth-order Darboux transformations will be done in several steps.

General setting. We consider the two generalized Schrödinger equations

$$\mathrm{i}\psi_t = \mathcal{H}\psi, \qquad \mathrm{i}\widetilde{\psi}_t = \widetilde{\mathcal{H}}\widetilde{\psi},$$

where the Hamiltonians $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are given in (2) and (4), respectively, and $\psi = \psi(x,t)$, $\widetilde{\psi} = \widetilde{\psi}(x,t)$ stand for the solutions. We seek an intertwiner $\mathcal{L}_n$ that satisfies the intertwining relation

$$\mathcal{L}_n(\mathrm{i}\partial_t - \mathcal{H}) = (\mathrm{i}\partial_t - \widetilde{\mathcal{H}})\mathcal{L}_n, \tag{45}$$

and we require this intertwiner to be a linear differential operator of order $n > 1$, that is

$$\mathcal{L}_n = \sum_{j=0}^{n} A_j\frac{\partial^j}{\partial x^j}.$$

Our task is to determine the functions $A_j = A_j(x,t)$ such that the latter operator satisfies (45).

The conventional case. Let us consider two conventional Schrödinger equations

$$\mathrm{i}\phi_t = H\phi, \qquad H = -\partial_{xx} + w, \tag{46}$$

$$\mathrm{i}\widetilde{\phi}_t = \widetilde{H}\widetilde{\phi}, \qquad \widetilde{H} = -\partial_{xx} + \widetilde{w}, \tag{47}$$

where $w = w(x, t)$ and $\widetilde{w} = \widetilde{w}(x, t)$ stand for the potentials. As was shown in [13], there is an intertwiner L_n of order n, satisfying the intertwining relation

$$L_n(\mathrm{i}\partial_t - H) = (\mathrm{i}\partial_t - \widetilde{H})L_n. \tag{48}$$

This intertwiner L_n can be given in an explicit form. To this end, let $v_1, v_2, \ldots, v_n$ be the solutions of the Schrödinger equation (46), such that the family $(v_1, v_2, \ldots, v_n, \phi)$ is linearly independent. Then the operator L_n satisfying (48) is given explicitly by

$$L_n(\cdot) = \beta \frac{W_{n+1,(v_j)}(\cdot)}{W_{n,(v_j)}}, \tag{49}$$

where $\beta = \beta(t)$ is arbitrary, $W_{n+1,(v_j)}$ and $W_{n,(v_j)}$ stand for the Wronskians of the families $(v_1, v_2, \ldots, v_n, \cdot)$ and $(v_1, v_2, \ldots, v_n)$, respectively. Finally, if the potentials w and $\widetilde{w}$ in Hamiltonians (46) and (47) satisfy

$$\widetilde{w} = w + \mathrm{i}\frac{\beta_t}{\beta} - 2\log\left(W_{n,(v_j)}\right)_{xx}, \tag{50}$$

then L is an intertwiner.

The point transformation. We now establish a connection between the conventional Schrödinger equation and its generalized counterpart (1), which we apply the following point transformation to:

$$\psi(x, t) = \exp[F(x, t)]\Phi[u(x, t), s(t)], \tag{51}$$

introducing a function $F = F(x, t)$ and new coordinates $u = u(x, t), s = s(t)$. This transformation converts the generalized Schrödinger equation to

$$\mathrm{i}\Phi_s + \left(\frac{u_x^2}{s_t h m}\right)\Phi_{uu} + \frac{1}{s_t h}\left(2\frac{F_x u_x}{m} - \frac{m_x u_x}{2m^2} + \frac{u_{xx}}{m} + \mathrm{i}hu_t\right)\Phi_u$$
$$+ \frac{1}{s_t h}\left(\mathrm{i}F_t h + \frac{F_x^2}{m} + F_{xx} - \frac{F_x m_x}{m^2} - V\right)\Phi = 0. \tag{52}$$

Here s_t denotes the derivative of s, note that s must not depend on x in order to preserve the linearity of the equation. Now we convert (52) into a conventional Schrödinger equation by requiring that the coefficient of Φ_{uu} is equal to 1 and that the coefficient of Φ_u vanishes:

$$\frac{u_x^2}{s_t h m} = 1, \qquad 2\frac{F_x u_x}{m} + g u_x + \frac{u_{xx}}{m} + \mathrm{i}hu_t = 0.$$

These conditions can be solved for the free parameters u and F of our point transformation (51):

$$u = \sqrt{s_t}\int \sqrt{hm}\,\mathrm{d}x, \tag{53}$$

$$F = -\int \left(\mathrm{i}\frac{hmu_t}{2u_x} - \frac{m_x}{2m} + \frac{u_{xx}}{2u_x}\right)\mathrm{d}x. \tag{54}$$

The new coordinate s remains arbitrary. Now, on plugging the settings (53) and (54) into equation (52), we obtain

$$\mathrm{i}\Phi_s + \Phi_{uu} + \frac{1}{s_t h}\left(\mathrm{i}F_t h + \frac{F_x^2}{m} + F_{xx} - \frac{F_x m_x}{m^2} - V\right)\Phi = 0, \tag{55}$$

where the explicit form of F is given in (54). Note that the coefficient of Φ in (55) is still written in the old coordinates x and t.

Transformation of the intertwining relation. Consider now the conventional intertwining relation (48) for the intertwiner L_n and potential difference as given in (49) and (50), respectively. This intertwining relation, applied to a solution ϕ of equation (46), has the following explicit form:

$$\frac{1}{W_{n,(v_j)}} W_{n+1,(v_j)}\left[(\mathrm{i}\partial_s + \partial_{uu} - w)\,\phi\right] = (\mathrm{i}\partial_s + \partial_{uu} - \widetilde{w})\,\frac{W_{n+1,(v_j)}(\phi)}{W_{n,(v_j)}}, \tag{56}$$

where $\widetilde{w}$ is given in (50). Note that we used the variables u and s instead of x and t, which will prove useful for the following reason: both sides of relation (56) contain a Schrödinger operator, corresponding to the conventional Schrödinger equations (46) and (47), respectively. Those equations are related to the generalized Schrödinger equations of the form (1) by means of the point transformation (51), (53) and (54). Thus, this point transformation connects the intertwining relation (56) with an intertwining relation that corresponds to the generalized Schrödinger equations. Let us now obtain this intertwining relation in an explicit form. To this end, we apply the inverse point transformation (51), (53) and (54) to (56), implying

$$\phi = \exp(-F)\psi, \tag{57}$$

$$v_j = \exp(-F)u_j, \tag{58}$$

where $u_1, u_2, \ldots, u_n$ are the solutions of the generalized Schrödinger equation (1). Furthermore, the derivatives in (56) need to be rewritten in terms of the variables x and t. These changes render the Wronskians in the following form [16]:

$$W_{n+1,(v_j)}(\phi) = \left(\frac{1}{u_x}\right)^{\frac{1}{2}n(n+1)} \exp\left[-(n+1)F\right] W_{n+1,(u_j)}(\psi), \tag{59}$$

$$W_{n,(u_j)} = \left(\frac{1}{u_x}\right)^{\frac{1}{2}n(n-1)} \exp\left(-nF\right) W_{n,(u_j)}, \tag{60}$$

where u_x stands for the derivative of the function u as given in (53). On plugging the settings (57)–(60) into the intertwiner L_n as given in (49), and changing its name to $\mathcal{L}_n$, we obtain

$$\begin{aligned}\mathcal{L}_n &= \beta\left(\frac{1}{u_x}\right)^n \frac{W_{n+1,(u_j)}(\cdot)}{W_{n,(u_j)}} \\ &= \beta\left(\frac{1}{hm}\right)^{\frac{n}{2}} \frac{W_{n+1,(u_j)}(\cdot)}{W_{n,(u_j)}},\end{aligned} \tag{61}$$

where in the last step we used the explicit form of u as given in (53). The Schrödinger operators in the intertwining relation (56) change into the generalized Schrödinger operators, corresponding to equation (1). It therefore remains to determine the potential w and its transformed counterpart (50) in the generalized Schrödinger equation. Without loss of generality, we can assume that the potential w in the intertwining relation (56) is given by

$$w = -\frac{1}{s_t h}\left(\mathrm{i}F_t h + \frac{F_x^2}{m} + F_{xx} - \frac{F_x m_x}{m^2} - V\right),$$

where the function F is defined in (54). This implies that the potential v in the generalized Schrödinger equation (1) reads $v = V$. According to (50), the transformed potential $\widetilde{v}$ in (4) must then read

$$\widetilde{v} = v + \mathrm{i}\frac{\beta_t}{\beta} - 2\log\left(W_{n,(v_j)}\right)_{uu}.$$

On employing the settings (57) and (58), and after rewriting the derivative in terms of the variables x and t, we obtain the following explicit potential difference:

$$\begin{aligned}\widetilde{v} &= v + \mathrm{i}s_t h\frac{\beta_t}{\beta} - 2s_t h\left\{\log\left[\exp(-nF)\left(\frac{1}{u_x}\right)^{\frac{1}{2}n(n-1)} W_{n,(u_j)}\right]\right\}_{uu}\\ &= v + \mathrm{i}s_t h\frac{\beta_t}{\beta} - 2s_t h\left[-nF + \frac{1}{2}n(n-1)\log\left(\frac{1}{mhs_t}\right) + \log\left(W_{n,(u_j)}\right)\right]_{uu}\\ &= v + \mathrm{i}s_t h\frac{\beta_t}{\beta} - \frac{2}{m}\left[-nF + \frac{1}{2}n(n-1)\log\left(\frac{1}{mhs_t}\right) + \log\left(W_{n,(u_j)}\right)\right]_{xx}\\ &\quad + 2h\left(\frac{1}{mh}\right)_x\left[-nF + \frac{1}{2}n(n-1)\log\left(\frac{1}{mhs_t}\right) + \log\left(W_{n,(u_j)}\right)\right]_x. \end{aligned} \tag{62}$$

We omit to substitute the explicit form of F as given in (54), since the resulting expression would become long and involved. We have now transformed the conventional intertwining relation (48) into a new intertwining relation for the generalized Schrödinger equation. This new relation has the following explicit form:

$$\mathcal{L}_n(\mathrm{i}\partial_t - \mathcal{H}) = (\mathrm{i}\partial_t - \widetilde{\mathcal{H}})\mathcal{L}_n. \tag{63}$$

Here, $\mathcal{L}_n$ is the nth-order intertwiner given in (61). Furthermore, the Hamiltonians $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are given in (2) and (4), respectively, and the difference between their potentials v and $\widetilde{v}$ is displayed in (62). The intertwining relation (63) is satisfied, as it was constructed from the well-known, conventional intertwining relation. Hence, we have shown that the higher order Darboux transformations for the generalized Schrödinger equation (1) can be constructed by means of intertwining relations, just as in the conventional case. Furthermore, since the explicit form of our Darboux transformation (61) and of the transformed potential (62) coincide with our previous results [17], it follows that both methods of construction—using point canonical transformations and intertwining relations—are equivalent.

The first-order case revisited. Note that in the first-order case our intertwining relation (63) must reduce to our former intertwining relation (3), where the intertwiner and the potential difference are given in (20) and (19), respectively. Let us now verify this, starting with the intertwiner (61) for $n = 1$. On evaluating (61), we get

$$\mathcal{L}_1 = \beta\sqrt{\frac{1}{hm}}\left(\frac{1}{u_1}\right)\det\begin{pmatrix} u_1 & 1\\ (u_1)_x & \partial_x\end{pmatrix} = \beta\sqrt{\frac{1}{hm}}\left[-\frac{(u_1)_x}{u_1} + \partial_x\right].$$

This coincides with (20), if we take into account that $K = -(u_1)_x/u_1$. Next, let us evaluate (62) for the case $n = 1$ and verify compatibility with (19). We get

$$\widetilde{v} = v - \mathrm{i}s_t h\frac{\beta_t}{\beta} - \frac{2}{m}\left[-F + \log(u_1)\right]_{xx} + 2h\left(\frac{1}{mh}\right)_x\left[-F + \log(u_1)\right]_x.$$

Inserting (54), setting $s(t) = t$ and simplifying gives the following result:

$$\widetilde{v} = v - \mathrm{i}h\left(\frac{h_t}{2h} + \frac{m_t}{2m} - \frac{\beta_t}{\beta}\right) + \frac{3h_x^2}{4h^2m} - \frac{3m_x^2}{4m^3} + \frac{m_x(u_1)_x}{m^2u_1} + \frac{2(u_1)_x^2}{mu_1^2} - \frac{h_{xx}}{2hm} + \frac{m_{xx}}{2m^2} - \frac{2(u_1)_{xx}}{hmu_1}.$$

This can be written in the form

$$\widetilde{v} = v + \mathrm{i}h\left[\frac{\beta_t}{\beta} - \frac{1}{2}\log{(hm)_t}\right] - 2\sqrt{\frac{h}{m}}\left[\frac{(u_1)_x}{u_1\sqrt{hm}}\right]_x - \sqrt{\frac{h}{m}}\left[\frac{1}{h}\left(\sqrt{\frac{h}{m}}\right)_x\right]_x,$$

and coincides with our result (19) if we take into account that $K = -(u_1)_x/u_1$.

6. Application

Let us now give a simple application of our formalism that shows how our Darboux transformation can be applied in a concrete case. To this end, we consider a generalized Schrödinger equation (1) of the following type:

$$\mathrm{i}h\psi_t = -\psi_{xx} + v\psi, \tag{64}$$

that is for a constant mass $m = 1$. The weight $h = h(x, t)$ and the potential $v = v(x)$ are chosen as follows:

$$h = \frac{p}{\alpha x}, \tag{65}$$

$$v = \frac{p}{x} - q^2, \tag{66}$$

where q and p are the real and positive constants, and $\alpha = \alpha(t)$ is an arbitrary function. A particular solution of (64) for the settings (65) and (66) is given by

$$\psi = \sin(qx)\exp\left(-\mathrm{i}\int \alpha\, \mathrm{d}t\right). \tag{67}$$

We will now perform the first-order Darboux transformation (20) on this function, taking as an auxiliary solution u_1 the function

$$u_1 = \cos(qx)\exp\left(-\mathrm{i}\int \alpha\, \mathrm{d}t\right). \tag{68}$$

Clearly, the two solutions (67) and (68) are linearly independent, as required for the Darboux transformation. After insertion of $m = 1$ and the functions (67), (68) into (20) and (19), we get the following results (note that $K = -(u_1)_x/u_1$):

$$\tilde{\psi} = \mathcal{L}\psi = \sqrt{\frac{\alpha}{p}x}\frac{\beta q}{\cos(qx)}\exp\left(-\mathrm{i}\int \alpha\, \mathrm{d}t\right)$$

$$\tilde{v} = \frac{p}{x} - q^2 + \frac{ip}{\alpha x}\left(\frac{\beta'}{\beta} - \frac{h_t}{2h}\right) + 2\sqrt{h}\left(\frac{K}{\sqrt{h}}\right)_x - \sqrt{h}\left[\frac{(\sqrt{h})_x}{h}\right]_x$$

$$= \frac{p}{x} - q^2 - \frac{1}{4x^2} + \frac{ip}{\alpha x}\left(\frac{\beta'}{\beta} + \frac{\alpha'}{2\alpha}\right) + 2\frac{q^2}{\cos^2(qx)} + \frac{q}{x}\tan(qx). \tag{69}$$

If we take $\beta = 1/\sqrt{\alpha}$ then the imaginary part of the transformed potential becomes zero. In fact, this is the simplest way to obtain a real-valued potential:

$$\tilde{v} = \frac{p}{x} - q^2 - \frac{1}{4x^2} + 2\frac{q^2}{\cos^2(qx)} + \frac{q}{x}\tan(qx). \tag{70}$$

This potential is stationary, as its initial counterpart v in (66). However, if we want the transformed potential (70) to be time dependent, then we need $\beta \neq 1/\sqrt{\alpha}$, since the time-dependent terms are in the potential's imaginary part. Finally, if we want the transformed potential (70) to be time dependent and real valued, then we need to set

$$\beta = \exp\left[\mathrm{i}\int\left(\gamma - \frac{\alpha'}{2\alpha}\right)\mathrm{d}t\right],$$

introducing a function $\gamma = \gamma(t)$, which renders the transformed potential (70) in the form

$$\tilde{v} = \frac{p}{x} - q^2 - \frac{1}{4x^2} + 2\frac{q^2}{\cos^2(qx)} + \frac{q}{x}\tan(qx) - \frac{p\gamma}{\alpha x}.$$

The fact that β has now turned complex does not matter, since it appears only in the transformed solution (69).

7. Concluding remarks

We have performed a thorough study of intertwining relations for the generalized Schrödinger equations of the form (1), which lead to the construction of arbitrary-order Darboux transformations and to a supersymmetry formulation. Both of the latter formalisms include all known special cases, e.g. for position-dependent mass [10] or for weighted energy [11]. Finally, we confirmed that constructing Darboux transformations via intertwining relations gives the same result as constructing them by means of point canonical transformations, as done in our previous paper [17].

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